BOUNDARY ACTIONS FOR AFFINE BUILDINGS AND HIGHER RANK CUNTZ-KRIEGER ALGEBRAS.

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ABSTRACT. Let Γ be a group of type rotating automorphisms of an affine building $\mathcal{B}$ of type $\widetilde{A}_2$. If Γ acts freely on the vertices of $\mathcal{B}$ with finitely many orbits, and if Ω is the (maximal) boundary of $\mathcal{B}$, then $C(\Omega) \rtimes \Gamma$ is a p.i.s.u.n. C^*-algebra. This algebra has a structure theory analogous to that of a simple Cuntz-Krieger algebra and is the motivation for a theory of higher rank Cuntz-Krieger algebras, which has been developed by T. Steger and G. Robertson. The K-theory of these algebras can be computed explicitly in the rank two case. For the rank two examples of the form $C(\Omega) \rtimes \Gamma$ which arise from boundary actions on $\widetilde{A}_2$ buildings, the two K-groups coincide.

INTRODUCTION

Two decades ago J. Cuntz and W. Krieger introduced the class of C^*-algebras which now bears their names [CK]. One reason for the importance of these algebras was their relationship to the classification of topological Markov chains. However they have also proved to be important in several other ways. Their theory has been refined and extended over the years by many authors from different points of view.

The Cuntz-Krieger algebra $\mathcal{O}_A$ associated with a nondegenerate $n \times n$ matrix A with entries in $\{0, 1\}$ is the universal C^*-algebra generated by partial isometries $s_1, \cdots, s_n$ satisfying

$$(0.1a) \qquad s_1 s_1^* + \cdots + s_n s_n^* = 1$$

$$(0.1b) \qquad s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*$$

Cuntz and Krieger proved that $\mathcal{O}_A$ is simple if and only if the matrix A is irreducible and not a permutation matrix. It was shown by M. Rørdam [Ror] that simple Cuntz-Krieger algebras are classified up to stable isomorphism by their K_0-group. The subsequent classification theorem of E. Kirchberg and C. Phillips [K1, K2, An] says that purely infinite, simple, separable, unital, nuclear (p.i.s.u.n.) C^*-algebras which satisfy the Universal Coefficient Theorem are classified up to isomorphism by their two K-groups together with the class of the identity element in K_0. This result applies in particular to simple Cuntz-Krieger algebras. The K-theory of a Cuntz-Krieger algebra $\mathcal{O}_A$ can be characterized as follows (see [C3]):

$$K_0(\mathcal{O}_A) = (\text{finite abelian group}) \oplus \mathbb{Z}^k; \; K_1(\mathcal{O}_A) = \mathbb{Z}^k.$$

The algebras $\mathcal{O}_A$ are therefore classified up to isomorphism by the group $K_0(\mathcal{O}_A)$ together with the class of the identity element in $K_0(\mathcal{O}_A)$.

Since p.i.s.u.n. C^*-algebras are now relatively well understood, it is of some interest when it happens that such algebras are naturally associated with concrete groups and geometries. Just such a situation has been studied by T. Steger and G. Robertson [RS1, RS2, RS3]. In [RS1] certain group actions on the boundaries of two dimensional buildings were investigated and the corresponding crossed product algebras were seen to be generated by two Cuntz-Krieger subalgebras. Subsequently [RS2] the properties of these geometric examples were abstracted to provide a set of

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axioms for a class of C^*-algebras. These algebras were seen to have a structure theory completely analogous to that of simple Cuntz-Krieger algebras. It is therefore appropriate to refer to them as higher rank Cuntz-Krieger algebras. The development of the theory in [RS2] was closely modeled on the original work of Cuntz and Krieger. The K-theory of rank two Cuntz-Krieger algebras was studied in [RS3]. The results are similar to the those of [C3] and depend upon the fact that a rank two Cuntz-Krieger algebra is stably isomorphic to the crossed product of an AF-algebra by a $\mathbb{Z}^2$-action. Suppose that Γ is a group of type rotating automorphisms of an affine building $\mathcal{B}$ of type $\widetilde{A}_2$, and that Γ acts freely on the vertices of $\mathcal{B}$ with finitely many orbits. Suppose that the algebra $\mathcal{A}$ arises from the boundary action of Γ. It follows from symmetry considerations that $K_0(\mathcal{A}) = K_1(\mathcal{A})$. If Γ also acts transitively on the vertices of $\mathcal{B}$ then the class in $K_0(\mathcal{A})$ of the identity element has torsion. Extensive computational results are given in [RS3].

1. Cuntz–Krieger algebras arising from boundary actions of free groups.

The geometric construction of higher rank Cuntz-Krieger algebras from group actions on affine buildings was motivated by work of J. Spielberg [Sp] in the rank one case. In [Sp] certain Cuntz-Krieger algebras were exhibited as crossed product algebras arising from actions of free products of cyclic groups on totally disconnected spaces. The construction has a particularly simple geometrical interpretation for a free group Γ of finite rank acting on its associated tree. The boundary of the tree is a totally disconnected space upon which Γ also acts and this action is used to define the relevant crossed product algebra.

Consider the specific group $\Gamma = \langle a, b \rangle$, the free group on two generators a and b. The homogeneous tree T of degree 4 is a Cayley graph of Γ. The vertices of T are the elements of Γ, i.e. reduced words in the generators and their inverses. The edges of T have the form (x, xs), where $x \in \Gamma$ and $s \in S = \{a, a^{-1}, b, b^{-1}\}$. It is convenient to label the directed edge (x, xs) by the generator s as in Figure 1.
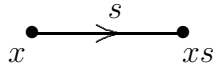

FIGURE 1. A labeled edge of the Cayley graph.

The *boundary* Ω of T can be identified with the set of all infinite reduced words $\omega = x_1 x_2 x_3 \ldots$, where $x_i \in S$. Ω has a natural compact totally disconnected topology in which a basic open neighbourhood of $\omega \in \Omega$ consists of those $\omega' \in \Omega$ whose corresponding infinite word agrees with that of ω on a finite initial segment. Left multiplication by $x \in \Gamma$ defines a homeomorphism of Ω and so induces an action α of Γ on $C(\Omega)$ by

$$\alpha(x)f(\omega) = f(x^{-1}\omega).$$

The crossed product $C(\Omega) \rtimes \Gamma$ is the universal C^*-algebra generated by $C(\Omega)$ and the image of a unitary representation π of Γ, satisfying the relations $\alpha(\gamma)f = \pi(\gamma)f\pi(\gamma)^*$ for $f \in C(\Omega)$ and $\gamma \in \Gamma$. It is convenient to write γ instead of $\pi(\gamma)$, thereby identifying elements of Γ with unitaries in $C(\Omega) \rtimes \Gamma$.

If $w = w(0)w(1)\ldots w(n) \in \Gamma$, where $w(i) \in S$, let $|w| = n$ and let $t(w) = w(n)$, the final letter of the reduced word w. For $w \in \Gamma$ let $\Omega(w)$ be the set of infinite words beginning with w (Figure 2). Then $\Omega(w)$ is open and closed in Ω and the sets $\Omega(w)$ for $w \in \Gamma$ form a basis for the topology of Ω. The boundary is partitioned into four parts according to the four possible initial letters of $\omega \in \Omega$ as shown in Figure 3. Denote by $p_w = \mathbf{1}_{\Omega(w)} \in C(\Omega)$ the characteristic function of $\Omega(w)$.
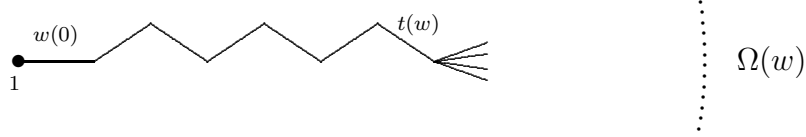
FIGURE 2. A basic open subset $\Omega(w)$ of the boundary.
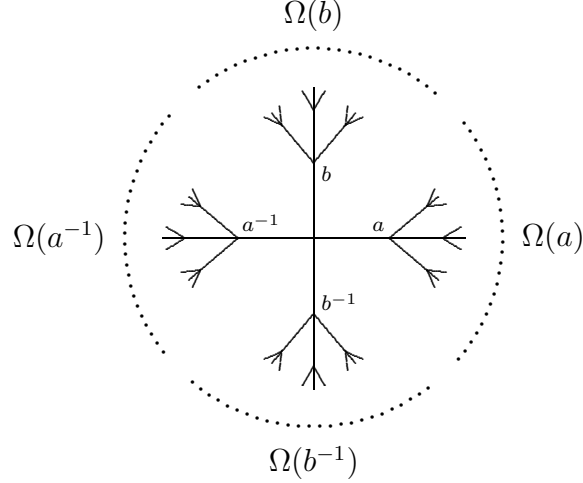
FIGURE 3. The homogeneous tree T of degree four.

If $u, v \in \Gamma$ and $t(u) = t(v)$, define $s_{u,v} = \gamma p_v \in C(\Omega) \rtimes \Gamma$, where $\gamma = uv^{-1}$. The covariance condition implies that $\gamma p_v = p_u \gamma$, so that $s_{u,v}$ is a partial isometry with initial projection p_v and final projection p_u.

Let $\mathcal{A}$ denote the C^*-subalgebra of $C(\Omega) \rtimes \Gamma$ generated by $\{s_{u,v} ; u, v \in \Gamma, t(u) = t(v)\}$. Then $\mathcal{A} = C(\Omega) \rtimes \Gamma$. To see this, firstly note that $\mathcal{A}$ contains $C(\Omega)$, since it contains $\{p_w ; w \in \Gamma\}$ and this set generates $C(\Omega)$ as a C^*-algebra. Also each element $u \in \Gamma$ lies in $\mathcal{A}$ since

$$u = \sum_{|x|=|u|+1} up_x = \sum_{|x|=|u|+1} s_{ux,x}.$$

Finally we claim that $\mathcal{A}$ is a Cuntz-Krieger algebra. For each $x \in S$ let

$$r_x = \sum_{y \in S; |xy|=2} s_{xy,y} = \sum_{y \in S; |xy|=2} xp_y.$$

Then

$$r_x r_x^* = \sum_{y \in S; |xy|=2} p_{xy} = p_x,$$

$$r_x^* r_x = \sum_{y \in S; |xy|=2} p_y = \sum_{y \in S; |xy|=2} r_y r_y^*.$$

Also

$$\sum_{x \in S} r_x r_x^* = \sum_{x \in S} p_x = \mathbf{1}.$$

Therefore $\{r_x; x \in S\}$ satisfies the classical Cuntz-Krieger relations (0.1).

For $u, v \in \Gamma$ with $t(u) = t(v)$, write

$$r_u = r_{u(0)} r_{u(1)} \ldots r_{t(u)} = \sum_{y \in S; |uy|=|u|+1} s_{uy,y}.$$

Then $r_u r_v^* = s_{u,v}$. Hence $\mathcal{A}$ is generated by $\{r_x; x \in S\}$. It follows that $C(\Omega) \rtimes \Gamma = \mathcal{A} = \mathcal{O}_M$, where

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

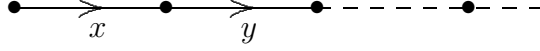A geometric interpretation of the condition $M(y,x) = 1$ is illustrated by Figure 4.

FIGURE 4. The condition $M(y,x) = 1$.

In the next section higher rank Cuntz-Krieger algebras are defined, based on analogues of the partial isometries $s_{u,v}$ rather than r_x.

2. HIGHER RANK CUNTZ-KRIEGER ALGEBRAS

We begin with some basic notation and terminology from [RS2]. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. Let $[m,n]$ denote $\{m, m+1, \ldots, n\}$, where $m \leq n$ are integers. If $m, n \in \mathbb{Z}^r$, say that $m \leq n$ if $m_j \leq n_j$ for $1 \leq j \leq r$, and when $m \leq n$, let $[m,n] = [m_1, n_1] \times \cdots \times [m_r, n_r]$. In $\mathbb{Z}^r$, let 0 denote the zero vector and let e_j denote the jth standard unit basis vector. We fix a finite set A which we refer to as an "alphabet".

A $\{0,1\}$-*matrix* is a matrix with entries in $\{0,1\}$. Choose nonzero $\{0,1\}$-matrices $M_1, M_2, \ldots, M_r$ and denote their elements by $M_j(b,a) \in \{0,1\}$ for $a, b \in A$. If $m, n \in \mathbb{Z}^r$ with $m \leq n$, let

$$W_{[m,n]} = \{w : [m,n] \to A; \ M_j(w(l+e_j), w(l)) = 1 \text{ whenever } l, l+e_j \in [m,n]\}.$$

Put $W_m = W_{[0,m]}$ if $m \geq 0$. Say that an element $w \in W_m$ has *shape* m, and write $\sigma(w) = m$. Thus W_m is the set of words of shape m, and we identify A with W_0 in the natural way. Define the initial and final maps $o : W_m \to A$ and $t : W_m \to A$ by $o(w) = w(0)$ and $t(w) = w(m)$.
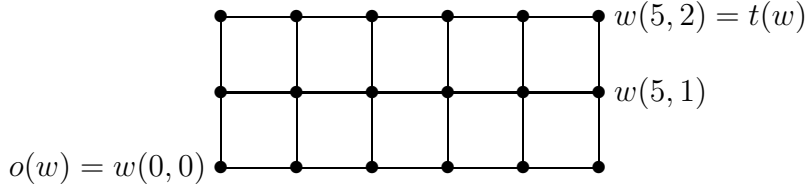

FIGURE 5. Representation of a two dimensional word of shape $m = (5, 2)$.

Fix a nonempty finite or countable set D (whose elements are "decorations"), and a map $\delta : D \to A$. Let $\overline{W}_m = \{(d, w) \in D \times W_m; \ o(w) = \delta(d)\}$, the set of "decorated words" of shape m, and identify D with $\overline{W}_0$ via the map $d \mapsto (d, \delta(d))$. Let $W = \bigcup_m W_m$ and $\overline{W} = \bigcup_m \overline{W}_m$, the sets of all words and all decorated words respectively. Define $o : \overline{W}_m \to D$ and $t : \overline{W}_m \to A$ by $o(d, w) = d$ and $t(d, w) = t(w)$. Likewise extend the definition of shape to $\overline{W}$ by setting $\sigma((d, w)) = \sigma(w)$.

Given $j \leq k \leq l \leq m$ and a function $w : [j, m] \to A$, define $w|_{[k,l]} \in W_{l-k}$ by $w|_{[k,l]} = w'$ where $w'(i) = w(i + k)$ for $0 \leq i \leq l - k$. If $\overline{w} = (d, w) \in \overline{W}_m$, define

$$\overline{w}|_{[k,l]} = w|_{[k,l]} \in W_{l-k} \text{ if } k \neq 0,$$

$$\text{and} \quad \overline{w}|_{[0,l]} = (d, w|_{[0,l]}) \in \overline{W}_l.$$

If $w \in W_l$ and $k \in \mathbb{Z}^r$, define $\tau_k w : [k, k+l] \to A$ by $(\tau_k w)(k+j) = w(j)$. If $w \in W_l$ where $l \geq 0$ and if $p \neq 0$, say that w is p-*periodic* if its p-translate, $\tau_p w$, satisfies $\tau_p w|_{[0,l] \cap [p, p+l]} = w|_{[0,l] \cap [p, p+l]}$.

Assume that the matrices M_i have been chosen so that the following conditions hold.

(H0): Each M_i is a nonzero $\{0, 1\}$-matrix.

(H1): Let $u \in W_m$ and $v \in W_n$. If $t(u) = o(v)$ then there exists a unique $w \in W_{m+n}$ such that

$$w|_{[0,m]} = u \qquad \text{and} \qquad w|_{[m,m+n]} = v.$$

We write $w = uv$ and say that the product uv exists. This product is clearly associative.

(H2): Consider the directed graph which has a vertex for each $a \in A$ and a directed edge from a to b for each i such that $M_i(b, a) = 1$. This graph is irreducible.

(H3): Let $p \in \mathbb{Z}^r$, $p \neq 0$. There exists some $w \in W$ which is not p-periodic.

Condition (H1) holds if the matrices M_i, $1 \leq i \leq r$ satisfy the following three conditions [RS2, Section 1].

(H1a): $M_i M_j = M_j M_i$.

(H1b): For $i < j$, $M_i M_j$ is a $\{0, 1\}$-matrix.

(H1c): For $i < j < k$, $M_i M_j M_k$ is a $\{0, 1\}$-matrix.

Definition 2.1. The C^*-algebra $\mathcal{A}$ is the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v}; \ u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

(2.1a) $$s_{u,v}{}^* = s_{v,u}$$

(2.1b) $$s_{u,v} s_{v,w} = s_{u,w}$$

(2.1c) $$s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw, vw}, \text{ for } 1 \leq j \leq r$$

(2.1d) $$s_{u,u} s_{v,v} = 0, \text{ for } u, v \in \overline{W}_0, u \neq v.$$

The partial isometry $s_{u,v}$ has initial projection $s_{v,v}$ and final projection $s_{u,u}$. If $\overline{W}_0 = D$ is finite then $\sum_{u \in \overline{W}_0} s_{u,u}$ is an identity for $\mathcal{A}$ [RS2, Section 3]. We refer to $\mathcal{A}$ as a higher rank Cuntz-Krieger algebra (of rank r) despite the fact that if $r = 1$ and D is infinite then $\mathcal{A}$ is in general only stably isomorphic to a classical Cuntz-Krieger algebra.

Suppose that $r = 1$, $M = M_1$, $D = A$ and δ is the identity map. Then the algebra $\mathcal{A}$ is isomorphic to the simple Cuntz-Krieger algebra $\mathcal{O}_M$. In fact $\mathcal{O}_M$ is generated by a set of partial isometries $\{S_a; a \in A\}$ satisfying the relations $S_a^* S_a = \sum_b M(a, b) S_b S_b^*$. If $u \in W$, let $S_u = S_{u(0)} S_{u(1)} \dots S_{t(u)}$. If $v \in W$ with $t(u) = t(v)$, define $S_{u,v} = S_u S_v^*$. Then the map $s_{u,v} \mapsto S_{u,v}$ establishes an isomorphism of $\mathcal{A}$ with $\mathcal{O}_M$. More generally $\mathcal{A}$ is isomorphic to a simple Cuntz-Krieger algebra whenever $r = 1$ and D is finite. A quick way to see this is to use [Ror, Proposition 6.6], together with the remarks following Theorem 2.2 below.

Tensor products of ordinary Cuntz-Krieger algebras can be identified as higher rank Cuntz-Krieger algebras $\mathcal{A}$. If $\mathcal{A}_1$, $\mathcal{A}_2$ are rank one Cuntz-Krieger algebras, with corresponding irreducible matrices M_1, M_2 indexed by alphabets A_1, A_2 then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the rank two Cuntz-Krieger algebra $\mathcal{A}$ arising from the pair of matrices $M_1 \otimes I, I \otimes M_2$ and the alphabet $A_1 \times A_2$. More interesting examples arise from group actions on affine buildings. We describe some of these later.

Theorem 2.2. [RS2] *The C^*-algebra $\mathcal{A}$ is purely infinite, simple and nuclear. Any nontrivial C^*-algebra with generators $S_{u,v}$ satisfying relations (2.1) is isomorphic to $\mathcal{A}$.*

If D is finite, then $\mathcal{A}$ is unital. Therefore $\mathcal{A}$ is a p.i.s.u.n. C^*-algebra and satisfies the Universal Coefficient Theorem [RS2]. By the Classification Theorem [K1, K2], $\mathcal{A}$ is classified by its K-groups and the class of the identity in K_0.

Denote by $\mathcal{A}_D$ the algebra $\mathcal{A}$ corresponding to a decorating set D. Recall that D is finite or countable. Given any set D of decorations we can obtain another set of decorations $D \times \mathbb{N}$, with the decorating map $\delta' : D \times \mathbb{N} \to A$ defined by $\delta'((d, i)) = \delta(d)$. It is shown in [RS2, Section 5] that

$\mathcal{A}_{D \times \mathbb{N}} \cong \mathcal{A}_D \otimes \mathcal{K}$. Also, for a fixed alphabet A and fixed transition matrices M_j, the isomorphism class of $\mathcal{A}_D \otimes \mathcal{K}$ is independent of D.

These facts are used in [RS2, Section 6] to prove that $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^r$, where $\mathcal{F}$ is an AF algebra. The algebra $\mathcal{F}$ is isomorphic to a subalgebra of $\mathcal{A}_{A \times \mathbb{N}}$, and is defined as an inductive limit algebra $\mathcal{F} = \varinjlim \mathcal{C}^{(m)}$ where $\mathcal{C}^{(m)}$ is an isomorphic copy of $\mathcal{C} = \bigoplus_{a \in A} \mathcal{K}$, with $\mathcal{K}$ the compact operators on a separable infinite dimensional Hilbert space. There is a commuting diagram of inclusions

$$
\begin{array}{ccc}
\mathcal{C}^{(m+e_k)} & \longrightarrow & \mathcal{C}^{(m+e_j+e_k)} \\
\uparrow & & \uparrow \\
\mathcal{C}^{(m)} & \longrightarrow & \mathcal{C}^{(m+e_j)}
\end{array}
$$

and the action of an element $l \in \mathbb{Z}^r$ on $\mathcal{F}$ maps the subalgebra $\mathcal{C}^{(m)}$ onto $\mathcal{C}^{(m+l)}$ for each $m \geq 0$.

3. A DECORATED RANK ONE EXAMPLE

We have seen that one use of decorating sets is to provide a method of passing from $\mathcal{A}$ to the stabilized algebra $\mathcal{A} \otimes \mathcal{K}$. On the other hand nontrivial decorating sets arise even in the rank one case when the construction of Section 1 is modified to take account of groups of automorphisms of a tree which act freely but not transitively on the vertices of the tree. Here is an example of how such a situation can arise.

Let X be a finite connected graph and let $\widetilde{X}$ be its universal covering graph. Let $\Gamma = \pi(X)$, the fundamental group of X. Then Γ is a free group which acts freely on $\widetilde{X}$ with finitely many vertex orbits [DD, Theorem I.9.1],[Ser, Chapter I Section 3]. Let $\partial \widetilde{X}$ be the boundary of the tree $\widetilde{X}$. Then Γ acts on $\partial \widetilde{X}$ and the crossed product algebra $C(\partial \widetilde{X}) \rtimes \Gamma$ is a rank one Cuntz-Krieger algebra. Let us look at a simple explicit example, where Γ is the free group on two generators.

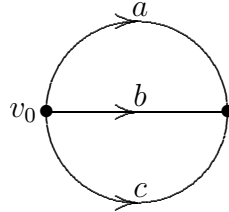Let X be the directed graph with two vertices and edges a, b, c, as illustrated in Figure 6.

FIGURE 6. The graph X.

Denote by $\overline{a}, \overline{b}, \overline{c}$ the opposite edges of a, b, c respectively. The universal covering graph $\widetilde{X}$ is a homogeneous tree of degree three (Figure 7).
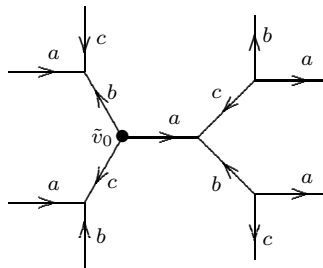

FIGURE 7. The universal covering graph $\widetilde{X}$.

By lifting paths in X with initial vertex v_0 to paths in $\widetilde{X}$ with initial vertex $\tilde{v}_0$, the labeling of the edges of X induces a labeling of the edges of $\widetilde{X}$. The boundary $\partial\widetilde{X}$ may be identified with the set of all semi-infinite geodesics in $\widetilde{X}$ beginning at $\tilde{v}_0$. The fundamental group Γ of X is the free group on two generators. Choose these generators to be the homotopy classes $[a\overline{b}], [a\overline{c}]$. Any path in X of even length based at v_0 is a loop and its homotopy class in Γ is a reduced word in $[a\overline{b}], [a\overline{c}]$ and their inverses. For example $[b\overline{c}] = [a\overline{b}]^{-1}[a\overline{c}]$.

The group Γ acts by left multiplication on the set of homotopy classes of paths in X beginning at v_0. Hence Γ acts on $\widetilde{X}$ and $\partial\widetilde{X}$. The crossed product algebra $C(\partial\widetilde{X}) \rtimes \Gamma$ is the rank one Cuntz-Krieger algebra $\mathcal{A}$ constructed in the following way. Let the alphabet be $A = \{a, \overline{a}, b, \overline{b}, c, \overline{c}\}$ and let the decorating set be $D = \{a, b, c\}$ with $\delta : D \to A$ the inclusion map. Let M be the matrix indexed by the elements of A, where $M(y, x) = 1$ if and only if xy is the labeling of a path of length 2 in X (that is, a simple loop). Then

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The set of decorated words $\overline{W} = \{(d, w) \in D \times W; \ o(w) = \delta(d)\}$ may clearly be identified with the set of words of the form $w(0)w(1)\ldots w(m)$, where $m \in \mathbb{Z}_+$, $w(0) \in D$, $w(i) \in A, 1 \le i \le m$ and $M(w(i+1), w(i)) = 1$. Thus $\overline{W}$ can be identified with the set of simple paths in X with initial vertex v_0. If $w \in \overline{W}$ then $t(w)$ corresponds to the final edge of the path.

The algebra $\mathcal{A}$ is now defined according to the procedure of Section 2. There is an isomorphism ϕ from $\mathcal{A}$ onto $C(\partial\widetilde{X}) \rtimes \Gamma$ defined as follows. For $u, v \in \overline{W}$ with $t(u) = t(v)$, let $[uv^{-1}]$ denote the element of Γ represented by the loop in X defined by the path u followed by the inverse path v^{-1}. Let $\Omega(v) \subset \partial\widetilde{X}$ be the characteristic function of the set of all boundary points which are represented by paths starting at the vertex $\tilde{v}_0$ and with initial segment covering the path v. Let $p_v = \mathbf{1}_{\Omega(v)} \in C(\partial\widetilde{X})$ be the characteristic function of $\Omega(v)$. Define $\phi(s_{u,v}) = [uv^{-1}]p_v$.

In this example and in that of Section 1 the group Γ is the same, but the actions are different. It turns out that the corresponding algebras have the same K-theory, namely $K_0 = K_1 = \mathbb{Z}^2$, and so the algebras are stably isomorphic. Everything above could also be expressed in the language of groupoids.

For completeness (and for later comparison) let us state the general one dimensional result. Assume that T is a tree with fixed base vertex O and boundary $\partial T = \Omega$. Let Γ be a group of automorphisms of T that acts freely on the vertex set with finitely many orbits. (In contrast to the example above, Γ need not be a free group.) Denote by $\mathcal{E}$ the set of edges of T and let $A = \Gamma \backslash \mathcal{E}$. Let $D = \{e \in \mathcal{E}; O \text{ is the initial vertex of } e\}$ and define an injective map $\delta : D \to A$ by $\delta(e) = \Gamma e$. Define a $\{0, 1\}$-matrix M by $M(b, a) = 1$ if and only if $a = \Gamma e_1, b = \Gamma e_2$ for edges e_1, e_2 in T lying as indicated in Figure 8. Let $\mathcal{A}_D$ be the rank one Cuntz-Krieger algebra constructed from these data.

Theorem 3.1. *With the above assumptions, $\mathcal{A}_D$ is isomorphic to $C(\Omega) \rtimes \Gamma$.*
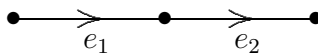

FIGURE 8. The geometric condition for $M(\Gamma e_2, \Gamma e_1) = 1$.

4. Affine Buildings

Let $\mathcal{B}$ be a finite dimensional simplicial complex, whose maximal simplices we shall call *chambers*. All chambers are assumed to have the same dimension and adjacent chambers have a common codimension one face. A *gallery* is a sequence of adjacent chambers. $\mathcal{B}$ is a *chamber complex* if any two chambers can be connected by a gallery. $\mathcal{B}$ is said to be *thin* if every codimension one simplex is a face of precisely two chambers. $\mathcal{B}$ is said to be *thick* if every codimension one simplex is a face of at least three chambers. A chamber complex $\mathcal{B}$ is called a *building* if it is the union of a family of subcomplexes, called *apartments*, satisfying the following axioms [Br2].

(B0): Each apartment Σ is a thin chamber complex with $\dim \Sigma = \dim \mathcal{B}$.
(B1): Any two simplices lie in an apartment.
(B2): Given apartments Σ, Σ' there exists an isomorphism $\Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.
(B3): $\mathcal{B}$ is thick.

Short readable introductions to the theory of buildings are provided by [Br2, Ca, St]. Detailed introductory texts are [Br1, G], while [Ron] is more advanced.

Proposition 4.1. [Br2] *The apartments in a building $\mathcal{B}$ are Coxeter complexes. The Coxeter group is called the Weyl group of $\mathcal{B}$.*

Proposition 4.2. [Br2] *If the apartments are infinite then the building is contractible. The apartments are then affine Coxeter complexes and the building is said to be affine.*

A building of type $\widetilde{A}_2$ has apartments which are all Coxeter complexes of type $\widetilde{A}_2$. Such a building is therefore a union of two dimensional apartments, each of which may be realized as a tiling of the Euclidean plane by equilateral triangles. From now on we shall consider only buildings of type $\widetilde{A}_2$. These are natural two dimensional analogues of homogeneous trees. In fact a homogeneous tree is a building of type $\widetilde{A}_1$. (Such a tree is contractible, its chambers are its edges and the apartments are complete geodesics.) Each vertex v of $\mathcal{B}$ is labeled with a *type* $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber has exactly one vertex of each type. An automorphism α of $\mathcal{B}$ is said to be *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices $v \in \mathcal{B}$.
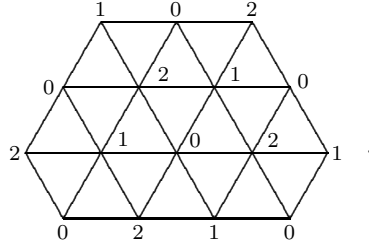

Figure 9. Part of an apartment showing vertex types.

A *sector* (or *Weyl chamber*) is a $\frac{\pi}{3}$-angled sector made up of chambers in some apartment (Figure 10). Two sectors are *equivalent* (or parallel) if their intersection contains a sector. (In a tree, sectors are semi-infinite geodesics.)

The boundary Ω of $\mathcal{B}$ is defined to be the set of equivalence classes of sectors in $\mathcal{B}$. In $\mathcal{B}$ fix some vertex O. For any $\omega \in \Omega$ there is a unique sector $[O, \omega)$ in the class ω having base vertex O [Ron, Theorem 9.6]. The boundary Ω is a totally disconnected compact Hausdorff space with a base for the topology given by sets of the form

$$\Omega(v) = \{\omega \in \Omega : [O, \omega) \text{ contains } v\}$$

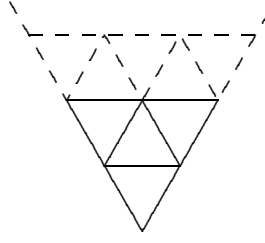where v is a vertex of $\mathcal{B}$ [CMS, Section 2].

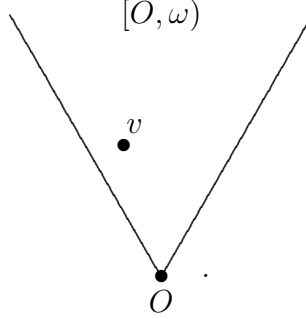FIGURE 10. A sector in a building $\mathcal{B}$ of type $\widetilde{A}_2$.

FIGURE 11. The sector $[O, \omega)$, where $\omega \in \Omega(v)$.

5. $\widetilde{A}_2$ GROUPS

Suppose that $\mathcal{B}$ is a building of type $\widetilde{A}_2$ and that Γ is a group of type rotating automorphisms of $\mathcal{B}$ which acts freely and transitively on the vertex set of $\mathcal{B}$. Such groups Γ are called $\widetilde{A}_2$ groups. They are good candidates to be rank two analogues of finitely generated free groups, which act in a similar way on buildings of type $\widetilde{A}_1$ (trees): an automorphism of a tree is automatically type rotating. The theory of $\widetilde{A}_2$ groups has been developed in detail in [CMSZ, CMS]. The $\widetilde{A}_2$ groups have a detailed combinatorial structure which makes them an ideal place to attack problems involving higher rank groups. For example [CMS] proved that $\widetilde{A}_2$ groups have Kazhdan's Property (T) and obtained exact Kazhdan constants, without the use of an embedding in any linear group. The $\widetilde{A}_2$ groups were the first examples of higher rank groups known to have property (RD) [RRS]. Also $\widetilde{A}_2$ groups are a natural place to prove higher rank analogues of results for von Neumann algebras associated with free groups [RR, RS].

The 1-skeleton of the building $\mathcal{B}$ is the Cayley graph of the $\widetilde{A}_2$ group Γ with respect to a canonical set P of $(q^2 + q + 1)$ generators, where q is a prime power. The set P may be identified with the set of points of a finite projective plane (P, L) of order q. There are $q^2 + q + 1$ points (elements of P) and $q^2 + q + 1$ lines (elements of L). Each point lies on $q + 1$ lines and each line contains $q + 1$ points. The set of lines L is identified with $\{x^{-1}; \ x \in P\}$. The relations satisfied by the elements of P are of the form $xyz = 1$. There is such a relation if and only if $y \in x^{-1}$: that is the point y is incident with the line x^{-1} in the projective plane (P, L). See Figure 12, which illustrates a typical chamber based at the identity element $1 \in \mathcal{B}$. As usual vertices are identified with elements of Γ and a directed edge of the form (a, as) with $a \in \Gamma$ is labeled by a generator $s \in P$.

If $q = 2$ there are eight $\widetilde{A}_2$ groups Γ, all of which embed as lattices in a linear group $\mathrm{PGL}(3, \mathbb{F})$ over a local field $\mathbb{F}$. If $q = 3$ there are 89 possible $\widetilde{A}_2$ groups, of which 65 have buildings which are not associated with linear groups [CMSZ].

Example 5.1. The group C.1 of [CMSZ], which we shall denote $\Gamma(\mathrm{C.1})$ has presentation

$$\langle x_i, 0 \le i \le 6 \,|\, x_0 x_0 x_6, x_0 x_2 x_3, x_1 x_2 x_6, x_1 x_3 x_5, x_1 x_5 x_4, x_2 x_4 x_5, x_3 x_4 x_6 \rangle.$$
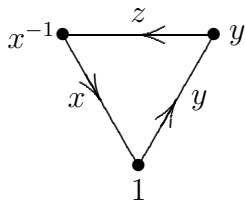
FIGURE 12. A chamber based at 1.

For this group $q = 2$, and there are $q^2 + q + 1 = 7$ generators. We have chosen this group as an example having the smallest possible number of generators (7) and relations (7). Moreover $\Gamma(\mathrm{C}.1)$ is a lattice subgroup of $G = \mathrm{PGL}(3, \mathbb{Q}_2)$, where $\mathbb{Q}_2$ is the field of 2-adic numbers [CMSZ]. The vertices of $\mathcal{B}$ can be identified with the discrete space G/K, where $K = \mathrm{PGL}(3, \mathbb{Z}_2)$, and $\mathbb{Z}_2$ is the ring of 2-adic integers. The boundary Ω of $\mathcal{B}$ can be identified with G/B where B is a minimal parabolic subgroup of G. Thus G acts naturally on the building $\mathcal{B}$ and its boundary. A detailed exposition of these facts from an analyst's point of view is given in [St].

Figure 13 illustrates the set of all fourteen neighbours of 1 in the Cayley graph of $\Gamma(\mathrm{C}.1)$. The fact that there is an edge between x_2 and x_0^{-1}, for example, is a consequence of the relation $x_0 x_2 x_3 = 1$, that is $x_2 x_3 = x_0^{-1}$. There are 21 edges in total, each lying in precisely one of the 21 chambers in $\mathcal{B}$ which contain the vertex 1. It is worth noting that although we have focused on the vertex 1, the set of nearest neighbours of any vertex in $\mathcal{B}$ also has the same structure of a finite projective plane.
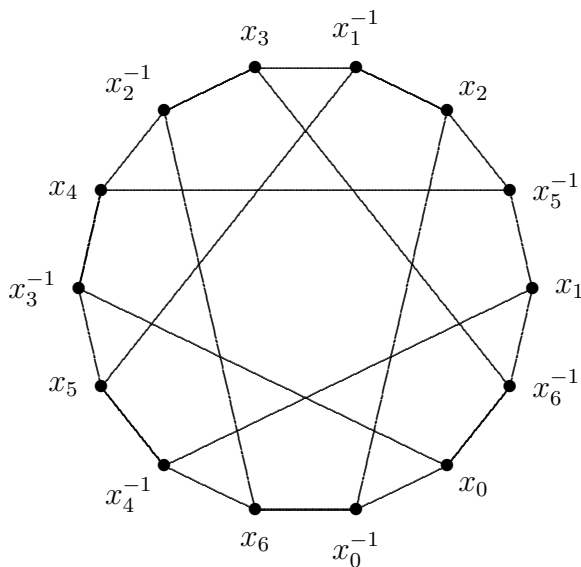
FIGURE 13. The projective plane of nearest neighbours of 1 for the group $\Gamma(\mathrm{C}.1)$.

6. ALGEBRAS ARISING FROM BOUNDARY ACTIONS ON $\widetilde{A}_2$ BUILDINGS

We are now in a position to describe the class of rank two Cuntz-Krieger algebras which provided the motivation for the general theory of [RS2]. Theorem 6.1 is a rank two version of the examples of Sections 1 and 3.

Theorem 6.1. [RS2, Theorem 7.7] *Let $\mathcal{B}$ be a building of type $\widetilde{A}_2$ with boundary Ω. Let Γ be a group of type rotating automorphisms of $\mathcal{B}$ that acts freely on the vertex set with finitely many orbits. Then there is an alphabet A, a decorating set D and matrices M_1, M_2 such that conditions (H0-H3) are satisfied and the corresponding rank two Cuntz-Krieger algebra $\mathcal{A}$ is isomorphic to $C(\Omega) \rtimes \Gamma$.*

For simplicity, consider the case where the action of Γ is also transitive on the vertex set, that is where Γ is an $\tilde{A}_2$ group, and the 1-skeleton of $\mathcal{B}$ is the Cayley graph of Γ relative to the generating set P. This is a two dimensional analogue of the situation described in Section 1 and the decorating set is trivial, i.e. $D = A$. For full generality, with a free but not necessarily transitive action of Γ on the vertices and a nontrivial decorating set, see [RS2]; compare also with Section 3.

Identify elements of Γ with vertices of the building $\mathcal{B}$. The alphabet A is defined to be the set of Γ-equivalence classes of basepointed parallelograms in $\mathcal{B}$, as illustrated in Figure 14. We refer to such an element of A as a tile. Each tile has a unique representative labelled parallelogram based at a fixed vertex as in Figure 14, where each edge label is a generator of Γ. The combinatorics of the finite projective plane (P, L) shows that there are precisely $q(q+1)(q^2+q+1)$ tiles $a \in A$.
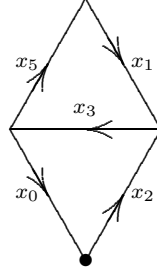

FIGURE 14. A tile $a \in A$ for the group $\Gamma(C.1)$.

Suppose that Γ is the group $\Gamma(C.1)$ of Example 5.1. Then $q = 2$ and $|A| = 42$. The transition matrices M_1, M_2 are defined as follows. If $a, b \in A$ we have $M_1(b, a) = 1$ if and only if there are labeled parallelograms representing a, b in the building $\mathcal{B}$ which lie as shown in Figure 15. In that diagram we have chosen edge labels representing specific choices for a, b. If no such diagram is possible then $M_1(b, a) = 0$. Figure 15 also illustrates the case $M_2(c, a) = 1$. Examination of the edge labels shows that we also have $M_2(b, a) = 1$ but that $M_1(c, a) = 0$. This geometric definition of the transition matrices is the exact analogue of the one dimensional situation described by Figure 4.
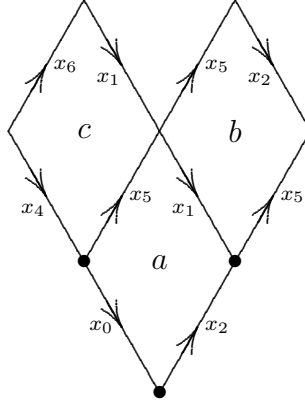

FIGURE 15. $M_1(b, a) = 1$, $M_2(c, a) = 1$.

Let $\mathfrak{p}$ be a parallelogram based at 1 in some apartment of $\mathcal{B}$. Then $\mathfrak{p}$ is a union of parallelograms representing tiles from the alphabet A (Figure 16). Associated to $\mathfrak{p}$ there is a two dimensional word $w = w(\mathfrak{p})$, as in Section 2. The map $\mathfrak{p} \mapsto w(\mathfrak{p})$ is bijective, and by abuse of notation we identify $\mathfrak{p}$ with $w(\mathfrak{p})$. For example in Figure 15, the two letters a, b define a word $w \in W_{(1,0)}$, with $w(0,0) = a$ and $w(1,0) = b$, whereas the two letters a, c define a word $w \in W_{(0,1)}$. If $w = w(\mathfrak{p}) \in W$, let the terminal letter $t(w) \in A$ be the tile of the parallelogram $\mathfrak{p}$ farthest from 1 (Figure 16). Also let $\Omega(w) = \{\omega \in \Omega; \mathfrak{p} \subset [1, \omega)\}$, the set of boundary points whose representative sectors based at 1 contain $\mathfrak{p}$.
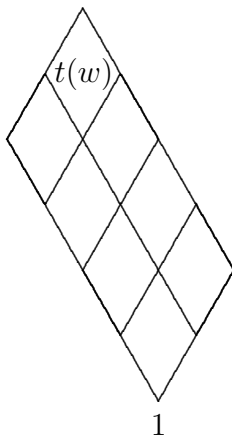
FIGURE 16. A parallelogram $\mathfrak{p}$ and the terminal letter of the word $w = w(\mathfrak{p})$.

We can now describe the isomorphism $\phi : \mathcal{A} \to C(\Omega) \rtimes \Gamma$. If $w_1, w_2 \in W$ with $t(w_1) = t(w_2) = a \in A$, let $\gamma \in \Gamma$ be the unique element such that $\gamma t(w_1) = t(w_2)$. Then

$$(6.1) \qquad \phi(s_{w_2,w_1}) = \gamma \mathbf{1}_{\Omega(w_1)} = \mathbf{1}_{\Omega(w_2)} \gamma.$$

This definition of $\phi(s_{w_2,w_1})$ is modeled on the rank one definition of $s_{u,v}$ given in Section 1. We refer the reader to [RS2, Section 7] for a proof of the isomorphism.

A vital step in [RS2, Section 7] is the verification of the conditions (H0-H3) needed to define the higher rank Cuntz-Krieger algebra $\mathcal{A}$ in Theorem 6.1. Condition (H0) is obvious. Condition (H1) follows from the fact that in the configuration illustrated by Figure 17, the tiles a, b, c determine a unique tile d lying in an apartment of $\mathcal{B}$ containing a, b, c. Condition (H3) follows from thickness of the building, which allows words to be extended so as to lack periodicities.
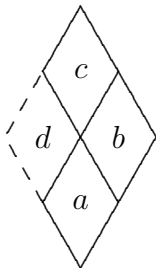

FIGURE 17. Tiles in an apartment.

The hardest condition to prove is (H2), i.e. irreducibility of the associated directed graph. This can be done by a direct combinatorial argument for $\widetilde{A}_2$ groups. If the group Γ of Theorem 6.1 is a lattice subgroup of $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field of characteristic zero, this is done in [RS2, Theorem 7.10] using the Howe-Moore Ergodicity Theorem. In forthcoming work of T. Steger it is shown how to extend the methods of the proof of the Howe–Moore Theorem and so prove Theorem 6.1 in the stated generality.

7. K-theory of rank 2 Cuntz-Krieger algebras

According to the Classification Theorem for p.i.s.u.n. algebras, a higher rank Cuntz-Krieger algebra $\mathcal{A}$ is classified by its K-groups and the class of the identity element in $K_0(\mathcal{A})$. It is therefore of some interest to compute the K-theory of these algebras. We have already observed that for a fixed alphabet A and fixed transition matrices M_j, the stable isomorphism class of $\mathcal{A}_D$ is independent of the decorating set D. For the purposes of computing $K_*(\mathcal{A})$, it is therefore enough to consider the algebra $\mathcal{A}$ with trivial $D = A$. One can then follow the original approach

to K-theory in the rank one case [C3], using the fact that $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^r$, where $\mathcal{F}$ is an AF algebra. A precise description of the K-theory was obtained in [RS3] for the case $r = 2$. For rank $r \geq 3$ the results are more complicated. We describe the rank 2 results of [RS3].

Let $\mathcal{A}$ be a rank two Cuntz-Krieger algebra associated with an alphabet A, trivial decorating set $D = A$ and matrices M_1, M_2, as in Section 2. The matrices $(I - M_1, I - M_2)$ and $(I - M_1^t, I - M_2^t)$ define homomorphisms $\mathbb{Z}^A \oplus \mathbb{Z}^A \to \mathbb{Z}^A$. The K-theory of $\mathcal{A}$ can be expressed as follows, where $\mathrm{tor}(G)$ denotes the torsion part of a finitely generated abelian group G, and $\mathrm{rank}(G)$ denotes the rank of G.

Proposition 7.1. [RS3, Proposition 2.14]

$$\mathrm{rank}(K_0(\mathcal{A})) = \mathrm{rank}(K_1(\mathcal{A})) = \mathrm{rank}(\mathrm{coker}\ {}_{(I-M_1,\ I-M_2)}) + \mathrm{rank}(\mathrm{coker}\ {}_{(I-M_1^t,\ I-M_2^t)})$$
$$\mathrm{tor}(K_0(\mathcal{A})) \cong \mathrm{tor}(\mathrm{coker}\ {}_{(I-M_1,\ I-M_2)})$$
$$\mathrm{tor}(K_1(\mathcal{A})) \cong \mathrm{tor}(\mathrm{coker}\ {}_{(I-M_1^t,\ I-M_2^t)}).$$

If the algebras arise from group actions on buildings we can say more.

Theorem 7.2. [RS3, Theorem 3.2] *Let Γ be a group of automorphisms of a building $\mathcal{B}$ of type $\widetilde{A}_2$ which acts freely on the set of vertices of $\mathcal{B}$ with finitely many orbits. Let Ω be the boundary of $\mathcal{B}$, so that $C(\Omega) \rtimes \Gamma$ is isomorphic to a higher rank Cuntz-Krieger algebra $\mathcal{A}$. If M_1, M_2 are the corresponding transition matrices, then*

$$K_0(\mathcal{A}) = K_1(\mathcal{A}) = \mathbb{Z}^{2n} \oplus \mathrm{tor}(\mathrm{coker}\ {}_{(I-M_1,\ I-M_2)})$$

where $n = \mathrm{rank}(\mathrm{coker}\ {}_{(I-M_1,\ I-M_2)})$.

The proof of Theorem 7.2 uses symmetry considerations in the building. Consider the special case where the action of Γ on the vertices of $\mathcal{B}$ is also transitive, i.e. Γ is an $\widetilde{A}_2$ group. Extensive computational results are given in [RS3] for more than 100 different groups with $2 \leq q \leq 11$, including all possible $\widetilde{A}_2$ groups for $q = 2, 3$. In particular, for the group $\Gamma = \Gamma(\mathrm{C.1})$ of Example 5.1, $K_0(C(\Omega) \rtimes \Gamma) = K_1(C(\Omega) \rtimes \Gamma) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})$ and $[\mathbf{1}] = 0$ in K_0. This example is not typical in that K_* usually has a free abelian component.

If Γ is an $\widetilde{A}_2$ group then $[\mathbf{1}]$ is always a torsion element of $K_0(C(\Omega) \rtimes \Gamma)$. In fact [RS3, Proposition 5.4] proves that $(q^2 - 1)[\mathbf{1}] = 0$. Moreover, for $q \not\equiv 1 \pmod 3$, $q - 1$ divides the order of $[\mathbf{1}]$ and for $q \equiv 1 \pmod 3$, $(q-1)/3$ divides the order of $[\mathbf{1}]$, [RS3, Proposition 5.5]. It follows that if Γ is an $\widetilde{A}_2$ group and $q \neq 2, 4$ then $[\mathbf{1}]$ is a nonzero torsion element in $K_0(C(\Omega) \rtimes \Gamma)$. Since $K_0 = K_1$ and the K_1 group of a rank one Cuntz-Krieger algebra is torsion free, it is immediate that for $q \neq 2, 4$, $C(\Omega) \rtimes \Gamma$ is not isomorphic to any rank one Cuntz-Krieger algebra.

Abundant experimental evidence suggests that for algebras associated with $\widetilde{A}_2$ groups it is always true that $[\mathbf{1}]$ has order $q - 1$ for $q \not\equiv 1 \pmod 3$ and has order $(q-1)/3$ for $q \equiv 1 \pmod 3$.

In view of Theorem 7.2, it is worth considering the general structure of rank two Cuntz-Krieger algebras $\mathcal{A}$ for which $K_0 = K_1$ and for which this group has even rank. The following more general result applies.

Proposition 7.3. *Let $\mathcal{A}$ be a p.i.s.u.n. C^*-algebra with $K_0(\mathcal{A}) = K_1(\mathcal{A}) = \mathbb{Z}^{2n} \oplus T$ where T is a finite abelian group. Then $\mathcal{A}$ is stably isomorphic to $\mathcal{A}_1 \otimes \mathcal{A}_2$, where $\mathcal{A}_1, \mathcal{A}_2$ are simple rank one Cuntz-Krieger algebras.*

Proof. By [Ror, Proposition 6.6] we can find simple rank one Cuntz-Krieger algebras $\mathcal{A}_1, \mathcal{A}_2$ such that $K_*(\mathcal{A}_1) = (\mathbb{Z}^n \oplus T, \mathbb{Z}^n)$ and $K_*(\mathcal{A}_2) = (\mathbb{Z}, \mathbb{Z})$. The Künneth Theorem for tensor products [Bl, Theorem 23.1.3] shows that $K_*(\mathcal{A}_1 \otimes \mathcal{A}_2) = (\mathbb{Z}^{2n} \oplus T, \mathbb{Z}^{2n} \oplus T)$. Since the algebras involved are all p.i.s.u.n. and satisfy the U.C.T., the result follows from the Classification Theorem [K2]. $\square$

In particular, the algebras of Theorem 7.2 are stably isomorphic to tensor products of rank one Cuntz-Krieger algebras.

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